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String versus Einstein frame in an AdS/CFT induced quantum dilatonic brane-world universe

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An AdS/CFT induced quantum dilatonic brane world where the 4D boundary is flat or a de Sitter (inflationary) or anti–de Sitter brane is considered. The classical brane tension is fixed but the boundary QFT produces the effective brane tension by means of the account of corresponding conformal anomaly induced effective action. This results in inducing brane worlds in accordance with the AdS/CFT setup as warped compactification. The explicit, independent construction of quantum induced dilatonic brane worlds in two frames, the string and Einstein frames, is done. Their complete equivalency is demonstrated for all quantum cosmological brane worlds under discussion, including several examples of classical brane-world black holes. This is different from quantum corrected 4D dilatonic gravity where a de Sitter solution exists in the Einstein but not in the Jordan (string) frame. The role of quantum corrections in massive graviton perturbations around an anti–de Sitter brane is briefly discussed.

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I. INTRODUCTION

Brane worlds are an alternative to standard Kaluza-Klein compactification. They naturally lead to the following nice features of multidimensional theory: trapping of 4D gravity on the brane [1], resolution of the hierarchy problem, and possibly the resolution of the cosmological constant problem. Different aspects of brane-world cosmology (for a very incomplete list of references see [2,3]) are under very active investigation.

The essential element of the original brane-world models is the presence in the theory of two free parameters (a bulk cosmological constant and brane tension, or brane cosmological constant). These parameters are fine-tuned (up to some extent) in order to construct a successful classical brane world. This is the most standard prescription which may be not completely satisfactory if one wishes to have a dynamical mechanism of brane tension origin.

From another side, one can fix the classical action on AdS-like space from the very beginning with the help of surface terms added in accordance with the AdS conformal field theory (CFT) correspondence [4]. Such terms should make the variational procedure well defined and also they should eliminate the leading divergence of the action. Brane tension is not considered as a free parameter anymore but it is fixed by the condition of the finiteness of spacetime when the brane goes to infinity. In this case, as parameters are fixed a consistent brane-world scenario is impossible, as a rule. However, other parameters may improve the situation when quantum effects are taken into account. Taking quantum CFT (including quantum gravity) on the brane one adds its contribution (the corresponding conformal anomaly induced effective action) to the total action. As a result, it changes the brane tension; the quantum induced brane world occurs as has been discovered in Refs. [5,6]. Actually, this represents the embedding of warped compactification (brane worlds) to the AdS/CFT correspondence; hence, one gets AdS/CFT induced quantum brane worlds [5,6] where the 4D boundary may be flat or de Sitter or anti–de Sitter spacetime. This is clearly the dynamical mechanism to get a curved brane world. It is easily generalized for the presence of a nontrivial dilaton; i.e., AdS/CFT induced quantum dilatonic brane worlds occur [7]. In other words, brane worlds are the consequence of the presence of quantum fields on the brane in accordance with the AdS/CFT setup. Moreover, such induced dilatonic brane worlds are even more related to the AdS/CFT correspondence as 5D dilatonic gravity represents the bosonic sector of 5D gauged supergravity (special parametrization). Even more, the dynamical determination of the 4D dilaton occurs.

In the study of quantum induced brane worlds, in the same way as for any other dilatonic gravity the following question appears: which frame to work with is the physical one? There are two convenient frames: the string (or Jordan) frame where scalar curvature explicitly couples with the dilaton and the Einstein frame where scalar curvature does not couple with the dilaton. Basically speaking, one should expect that results obtained in these two frames are not equivalent.

Indeed, in quantum field theory (QFT) the choice of different variables and (or) form of action corresponds to different parametrizations. QFT results are parametrization dependent; only the S matrix is gauge and parametrization independent. [Even the quantization procedure (for review, see [8]) is parametrization dependent.] As usually the consideration is one loop, one should expect in many cases an explicit parametrization dependence. Moreover, it is known

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that even for classical dilatonic gravity a (singular) solution may exist in only one parametrization. Hence, the question of frame dependence should be carefully analyzed for all solutions at hand. This is the main purpose of the present work: to compare string frame quantum induced dilatonic brane worlds with their analogues in the Einstein frame.

In the next section as a simple example, 4D dilatonic (Brans-Dicke) theory with large N quantum spinor corrections is considered. In the Einstein frame where the spinor is the dilaton coupled one a de Sitter universe solution with decaying dilaton exists. Working with the same theory in the string (Jordan) frame where the spinor becomes minimal, one finds that the above solution does not exist. Hence, it is shown that two frames in 4D dilatonic gravity with quantum corrections are not equivalent.

In Sec. III we consider the 5D dilatonic gravity action with 4D boundary terms induced by a conformal anomaly of the brane, the dilaton coupled spinor. Explicit examples of de Sitter, flat, and anti–de Sitter dilatonic branes are constructed in the Einstein frame. The dynamical mechanism to determine the dilaton on the brane is presented. In Sec. IV the same investigation is done in the string frame. The brane spinor is now minimal. The same AdS/CFT induced quantum brane worlds are proved to exist. Hence, for quantum corrected cosmological dilatonic brane worlds one has the equivalence of string and Einstein frames.

In Sec. V the equivalency of the string and Einstein frames is demonstrated for a number of classical dilatonic brane-world black holes. In Sec. VI some remarks on the massive graviton modes around the dilatonic AdS_4 brane are made. The role of brane quantum corrections for massive graviton modes is clarified. A brief summary and some outlook are given in the final section.

II. JORDAN AND EINSTEIN FRAMES FOR 4D QUANTUM CORRECTED DILATONIC GRAVITY

In the study of dilatonic gravities the interesting question appears, which frame among the few possible ones is the physical one? Basically speaking, there are two convenient frames to work with: the string (or Jordan) frame and the Einstein frame. These two are related by conformal transformation. The best known example is provided by standard Brans-Dicke theory (with matter). The four-dimensional action in the Jordan frame is

$$S_{BD}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R-\frac{\omega}{\phi}(\nabla_\mu\phi)(\nabla^\mu\phi)\right]+S_M, \tag{1}$$

where ϕ is the Brans-Dicke (dilaton) field with ω being the coupling constant and S_M is the matter action.

Performing the following conformal transformation and a redefinition of the scalar field,

$$\tilde{g}_{\mu\nu}=G\phi g_{\mu\nu},\quad \tilde{\phi}=\sqrt{\frac{2\omega+3}{16\pi G}}\ln(G\phi),\quad 2\omega+3>0, \tag{2}$$

one gets the action in the Einstein frame:

$$S=\int d^4x\sqrt{-\tilde{g}(x)}\left[\frac{\tilde{R}}{16\pi G}-\frac{1}{2}(\tilde{\nabla}_\mu\tilde{\phi})(\tilde{\nabla}^\mu\tilde{\phi})+\exp(A\tilde{\phi})L_M(\tilde{g})\right], \tag{3}$$

where $A=-8[\sqrt{\pi G/(2\omega+3)}]$. It is expected that these two actions (at least for regular solutions) should lead to equivalent results. However, an explicit consideration shows that it is not always so (for a review, see [9]). That is why it was argued in Ref. [9] that it is the Einstein frame which is the physical one. Of course, such a state of affairs is not satisfactory.

In quantum field theory the choice of different variables corresponds to different parametrizations. It is known that generally speaking it leads to parametrization dependent results: it is only the S matrix which should be the same in different parametrizations. Of course, this should be true only in a complete theory where an account of all loops is taken. As usually the consideration is one loop, one should expect a parametrization dependence already at one loop.

Let us consider an explicit example in the Einstein frame where quantum corrections are taken into account. As a matter Lagrangian we take the one associated with N massless (Dirac) spinors, i.e., $L_M=\Sigma_{i=1}^N\bar{\psi}_i\gamma^\mu\nabla_\mu\psi^i$. There is no problem to add other types of matter (say, scalar or vector fields). The above choice is made only for the sake of simplicity.

We shall make use of the effective action (EA) formalism (for an introduction, see [10]). The corresponding 4D anomaly induced EA for dilaton coupled scalars, vectors, and spinors has been found in Ref. [11].

Hence, starting from the theory with the action (no classical background spinors)

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{16\pi G}-\frac{1}{2}(\nabla_\mu\phi)(\nabla^\mu\phi)+\exp(A\phi)\sum_{i=1}^N\bar{\psi}_i\gamma^\mu\nabla_\mu\psi^i\right], \tag{4}$$

we will discuss Friedmann-Robertson-Walker (FRW-) type cosmologies:

$$ds^2=-dt^2+a(t)^2\,dl^2, \tag{5}$$

where dl^2 is the line metric element of a three-dimensional flat space.

The computation of the anomaly induced EA for the dilaton coupled spinor field has been done in [11], and the result, in a noncovariant local form, reads

$$W=\int d^4x\sqrt{-\bar{g}}\left\{b\bar{F}\sigma_1+2b'\sigma_1\left[\bar{\Box}^2+2\bar{R}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu-\frac{2}{3}\bar{R}\bar{\Box}+\frac{1}{3}(\bar{\nabla}^\mu\bar{R})\bar{\nabla}_\mu\right]\sigma_1+b'\sigma_1\left(\bar{G}-\frac{2}{3}\bar{\Box}\bar{R}\right)-\frac{1}{18}(b+b')[\bar{R}-6\bar{\Box}\sigma_1-6(\bar{\nabla}_\mu\sigma_1)(\bar{\nabla}^\mu\sigma_1)]^2\right\}, \tag{6}$$

where $\sigma_1=\sigma+A\phi/3$, the square of the Weyl tensor is given by $F=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2R_{\mu\nu}R^{\mu\nu}+(1/3)R^2$, and the Gauss-Bonnet invariant is $G=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2$. For Dirac spinors $b=3N/60(4\pi)^2$, $b'=-11N/360(4\pi)^2$.

Then we find the following Einstein frame, quantum corrected solution whose metric is expressed in the Jordan frame as

$$ds_J^2=a_J^2(\eta)(-d\eta^2+dl^2),$$

$$a_J^2(\eta)=\exp\left(-\frac{\phi}{\sqrt{\frac{2\omega+3}{16G}}}\right)a^2(\eta)=a_0\eta^{-2\zeta},$$

$$\zeta\equiv\frac{1}{2H_1}\sqrt{\frac{16\pi G}{2\omega+3}}+1$$

$$=\frac{1}{\sqrt{2\omega+3}\left\{-\frac{3}{16}\sqrt{2\omega+3}\pm\sqrt{\frac{9}{256}(2\omega+3)-\frac{1}{6}}\right\}}+1$$

$$=-\frac{1}{8}\mp\sqrt{\frac{81}{64}-\frac{6}{2\omega+3}}. \tag{7}$$

Here a_0 is an arbitrary constant. On the other hand, one finds the dilaton field ϕ_J in the Jordan frame as

$$\phi=\phi_0\eta^{(1/H_1)[\sqrt{16\pi G/(2\omega+3)}]}=\phi_0\eta^{2(\zeta-1)},\qquad \phi_0=\frac{1}{a_0G}. \tag{8}$$

Let us analyze the equations of motion in the Jordan frame (for the form of the transformation to the string frame see Sec. V). The variations over ϕ and σ give the following equations:

$$0=6(\sigma''+\sigma'^2)e^{2\sigma}-\frac{\omega\phi'^2}{\phi^2}e^{2\sigma}-2\omega\left(\frac{\phi' e^{2\sigma}}{\phi}\right), \tag{9}$$

$$0=\frac{2}{16\pi}\left(6(\sigma''+\sigma'^2)+\frac{\omega\phi'^2}{\phi}\right)e^{2\sigma}+\frac{6(e^{2\sigma})''-12(\sigma' e^{2\sigma})'}{16\pi}+4b'\sigma''''-4(b+b')\times\{(\sigma''-\sigma'^2)''+2(\sigma'(\sigma''-\sigma'^2))'\}. \tag{10}$$

Here $'\equiv d/d\eta$. We can check that the solution (7) and (8) does not satisfy Eq. (9). If the solution in the Jordan frame were equivalent to that in the Einstein frame even at the quantum level, we should have $\sigma_1=\sigma_J\equiv\ln a_J$ but we have $\sigma_1=\sigma+A\phi/3=\sigma-4/3\ln G\phi_J$ and $\sigma_J=\sigma-1/2\ln G\phi_J$. This is the origin of the inequivalence. Thus, it is demonstrated that for the universe model under consideration the Jordan and Einstein frames in 4D dilatonic gravity with quantum corrections are not equivalent. Different parametrizations lead to different results (parametrization choice dependence). The physical results are expected to be the same only for the S matrix in the full theory (nonperturbative regime).

III. INFLATIONARY DILATONIC BRANE-WORLD UNIVERSE IN THE EINSTEIN FRAME

In this section we present a review of quantum induced dilatonic brane worlds found in Ref. [7]. The model is discussed in the Einstein frame and using Euclidean notation. This scenario represents an extension to the nonconstant dilaton case of the earlier scenario of Refs. [5,6] where quantum brane worlds were realized in frames of the AdS/CFT correspondence by adding quantum CFT on the brane to the effective action.

We start with the Euclidean signature action S which is the sum of the Einstein-Hilbert action $S_{\rm EH}$ including the dilaton ϕ with potential $V(\phi)=12/l^2+\Phi(\phi)$, the Gibbons-Hawking surface term $S_{\rm GH}$, and the surface counterterm S_1,[1]

$$S=S_{\rm EH}+S_{\rm GH}+2S_1, \tag{11}$$

$$S_{\rm EH}=\frac{1}{16\pi G}\int d^5x\sqrt{g_{(5)}}\left(R_{(5)}-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi+\frac{12}{l^2}+\Phi(\phi)\right), \tag{12}$$

$$S_{\rm GH}=\frac{1}{8\pi G}\int d^4x\sqrt{g_{(4)}}\nabla_\mu n^\mu, \tag{13}$$

$$S_1=-\frac{1}{16\pi G}\int d^4\sqrt{g_{(4)}}\left(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\right). \tag{14}$$

Here the quantities in the five-dimensional bulk spacetime are specified by the suffices (5) and those in the boundary four-dimensional spacetime are specified by (4). The factor of 2 in front of S_1 in Eq. (11) is coming from the fact that we have two bulk regions which are connected with each other by the brane. It is clear that the above representation corresponds to the Einstein frame. In Eq. (13), n^μ is the unit vector normal to the boundary.

A. Bulk solutions

In this subsection, we find some explicit solutions in the bulk space.

We now assume a metric in the form

[1]We use the following curvature conventions:

$$R=g^{\mu\nu}R_{\mu\nu},$$
$$R_{\mu\nu}=R^\lambda{}_{\mu\lambda\nu},$$
$$R^\lambda{}_{\mu\rho\nu}=-\Gamma^\lambda_{\mu\rho,\nu}+\Gamma^\lambda_{\mu\nu,\rho}-\Gamma^\eta_{\mu\rho}\Gamma^\lambda_{\nu\eta}+\Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\rho\eta},$$
$$\Gamma^\eta_{\mu\lambda}=\frac{1}{2}g^{\eta\nu}(g_{\mu\nu,\lambda}+g_{\lambda\nu,\mu}-g_{\mu\lambda,\nu}).$$

$$ds^2=f(y)dy^2+y\sum_{i,j=0}^{3}\hat{g}_{ij}(x^k)dx^i\,dx^j, \tag{15}$$

and ϕ depends only on y: $\phi=\phi(y)$. Here $\hat{g}_{ij}$ is the metric of the Einstein manifold, which is defined by $r_{ij}=k\hat{g}_{ij}$, where r_{ij} is the Ricci tensor constructed with $\hat{g}_{ij}$ and k is a constant. Then we obtain the following equations of motion in the bulk:

$$0=\frac{3}{2y^2}-\frac{2kf}{y}-\frac{1}{4}\left(\frac{d\phi}{dy}\right)^2-\left(\frac{6}{l^2}+\frac{1}{2}\Phi(\phi)\right)f, \tag{16}$$

$$0=\frac{d}{dy}\left(\frac{y^2}{\sqrt{f}}\frac{d\phi}{dy}\right)+\Phi'(\phi)y^2\sqrt{f}. \tag{17}$$

It is convenient to introduce the new coordinate z:

$$z=\int dy\sqrt{f(y)}. \tag{18}$$

By solving y with respect to z, we obtain the warp factor $l^2e^{2\hat{A}(z,k)}=y(z)$. Here one assumes the metric of five-dimensional spacetime as follows:

$$ds^2=dz^2+e^{2A(z,\sigma)}\tilde{g}_{\mu\nu}\,dx^\mu\,dx^\nu,$$

$$\tilde{g}_{\mu\nu}\,dx^\mu\,dx^\nu\equiv l^2(d\sigma^2+d\Omega_3^2), \tag{19}$$

where $d\Omega_3^2$ corresponds to the metric of a three-dimensional unit sphere. Suppose that $A(z,\sigma)$ can be decomposed into the sum of a z-dependent part $\hat{A}(z)$ and a σ-dependent part and therefore $l^2e^{2\hat{A}(z)}\hat{g}_{\mu\nu}=e^{2A(z,\sigma)}\tilde{g}_{\mu\nu}$. Then, for the unit sphere $(k=3)$,

$$A(z,\sigma)=\hat{A}(z,k=3)-\ln\cosh\sigma, \tag{20}$$

for the flat Euclidean space $(k=0)$,

$$A(z,\sigma)=\hat{A}(z,k=0)+\sigma, \tag{21}$$

and for the unit hyperboloid $(k=-3)$,

$$A(z,\sigma)=\hat{A}(z,k=-3)-\ln\sinh\sigma. \tag{22}$$

When $\Phi(\phi)=0$, there exists the following AdS-like solution of the equations of motion [12]:

$$ds^2=f(y)dy^2+y\sum_{i,j=0}^{d-1}\hat{g}_{ij}(x^k)dx^i\,dx^j,$$

$$f=\frac{d(d-1)}{4y^2\lambda^2\left(1+\frac{c^2}{2\lambda^2y^d}+\frac{kd}{\lambda^2y}\right)},$$

$$\phi=c\int dy\sqrt{\frac{d(d-1)}{4y^{d+2}\lambda^2\left(1+\frac{c^2}{2\lambda^2y^d}+\frac{kd}{\lambda^2y}\right)}}. \tag{23}$$

Here $\lambda^2=12/l^2$.

When $\Phi(\phi)\neq 0$, by using Eqs. (16) and (17), one can delete f from the equations and we obtain an equation that contains only the dilaton field ϕ:

$$0=\left\{\frac{5k}{2}-\frac{k}{4}y^2\left(\frac{d\phi}{dy}\right)^2+\left[\frac{3}{2}y-\frac{y^3}{6}\left(\frac{d\phi}{dy}\right)^2\right]\left(\frac{6}{l^2}+\frac{1}{2}\Phi(\phi)\right)\right\}$$
$$\times\frac{d\phi}{dy}+\frac{y^2}{2}\left(\frac{2k}{y}+\frac{6}{l^2}+\frac{1}{2}\Phi(\phi)\right)\frac{d^2\phi}{dy^2}$$
$$+\left[\frac{3}{4}-\frac{y^2}{8}\left(\frac{d\phi}{dy}\right)^2\right]\Phi'(\phi). \tag{24}$$

We now consider a solvable case where

$$\frac{6}{l^2}+\frac{1}{2}\Phi(\phi)=-\frac{2k}{y}. \tag{25}$$

The explicit form, or ϕ dependence, of $\Phi(\phi)$ can be determined after solving the equations of motion such as the following:

$$\phi=\pm\sqrt{6}\ln(m^2y). \tag{26}$$

Here m^2 is a constant of integration. The explicit form of $\Phi(\phi)$ is

$$\Phi(\phi)=-\frac{12}{l^2}-4km^2e^{\mp\phi/\sqrt{6}}. \tag{27}$$

One can also find that Eq. (16) is trivially satisfied. Integrating Eq. (17), we obtain

$$f=\frac{1}{-\frac{2ky}{9}+\frac{f_0}{y^2}}. \tag{28}$$

Here f_0 is a constant of integration and f_0 should be positive in order that f be positive for large y. There is a (curvature) singularity at $y=0$. One should also note that when $k>0$, a horizon appears at

$$y^3=y_0^3\equiv\frac{9f_0}{2k} \tag{29}$$

and we find

$$y\leqslant y_0. \tag{30}$$

B. Brane solutions

In this subsection, we investigate if there is a solution with a brane including the quantum correction from N massless brane Majorana spinors coupled with the dilaton. For simplicity, we consider only the case that the potential is constant.

On the brane, one obtains the following equations by the variations over A and ϕ:

$$0=\frac{48l^4}{16\pi G}\left(\partial_z A-\frac{1}{l}-\frac{l}{24}\Phi(\phi)\right)e^{4A}, \tag{31}$$

$$0=-\frac{l^4}{8\pi G}e^{4A}\partial_z\phi-\frac{l^5}{32\pi G}e^{4A}\Phi'(\phi). \tag{32}$$

With Eq. (27) and the solution (28), these equations are

$$0=\frac{1}{2R^2}\sqrt{\frac{f_0}{R^4}-\frac{2kR^2}{9}}-\frac{1}{2l}+\frac{kl}{3R^2}, \tag{33}$$

$$0=\sqrt{\frac{f_0}{R^4}-\frac{2kR^2}{9}}+\frac{kl}{kl}. \tag{34}$$

Here we assume that the brane lies at $y=y_0$ or $z=z_0$. The radius R of the brane is defined by $R=e^{\hat{A}(z_0)}$. Equation (34) tells that $k\leqslant 0$ but by combining Eqs. (33) and (34), we find $R^2=kl^2/2$. Then there is no consistent classical solution.

We now consider the case that the matter on the brane is some QFT such as QED or QCD. Of course, such a theory is classically a conformally invariant one. As an explicit example, in order to be able to apply a large-N expansion we suppose that the dominant contribution is due to N massless Majorana spinors coupled with the dilaton, whose action is given by

$$S=\int\sqrt{g_{(4)}}e^{a\phi}\sum_{i=1}^{N}\bar{\Psi}_i\gamma^\mu D_\mu\Psi_i. \tag{35}$$

The case of minimal spinor coupling corresponds to the choice $a=0$. Note that from Brans-Dicke theory one knows that for the Einstein frame the nonminimal dilaton coupling with the matter is the typical case. Then the trace anomaly induced action W has the following form [11]:

$$W=b\int d^4x\sqrt{\tilde{g}}\tilde{F}A_1+b'\int d^4x\Bigg\{A_1\Bigg[2\tilde{\Box}^2+\tilde{R}_{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu-\frac{4}{3}\tilde{R}\tilde{\Box}^2+\frac{2}{3}(\tilde{\nabla}^\mu\tilde{R})\tilde{\nabla}_\mu\Bigg]A_1+\left(\tilde{G}-\frac{2}{3}\tilde{\Box}\tilde{R}\right)A_1\Bigg\}-\frac{1}{12}\left\{b''+\frac{2}{3}(b+b')\right\}\times\int d^4x[\tilde{R}-6\tilde{\Box}A_1-6(\tilde{\nabla}_\mu A_1)(\tilde{\nabla}^\mu A_1)]^2. \tag{36}$$

Here

$$A_1=A+\frac{a\phi}{3} \tag{37}$$

and

$$b=\frac{3N}{60(4\pi)^2},\quad b'=-\frac{11N}{360(4\pi)^2}. \tag{38}$$

We also choose $b''=0$ as it may be changed by a finite renormalization of the classical gravitational action. In Eq. (36), one chooses the four-dimensional boundary metric as

$$g_{(4)\mu\nu}=e^{2A}\tilde{g}_{\mu\nu}, \tag{39}$$

and we specify the quantities given by $\tilde{g}_{\mu\nu}$ by using a tilde. $G(\tilde{G})$ and $F(\tilde{F})$ are the Gauss-Bonnet invariant and the square of the Weyl tensor, which are given as

$$G=R^2-4R_{ij}R^{ij}+R_{ijkl}R^{ijkl},$$

$$F=\frac{1}{3}R^2-2R_{ij}R^{ij}+R_{ijkl}R^{ijkl}. \tag{40}$$

For simplicity, we consider a constant potential $[\Phi(\phi)=0]$ case. Then the brane equations are

$$0=\frac{48l^4}{16\pi G}\left(\partial_z A-\frac{1}{l}\right)e^{4A}+b'(4\partial_\sigma^4A_1-16\partial_\sigma^2A_1)-4(b+b')(\partial_\sigma^4A_1+2\partial_\sigma^2A_1-6(\partial_\sigma A_1)^2\partial_\sigma^2A_1), \tag{41}$$

$$0=-\frac{l^4}{8\pi G}e^{4A}\partial_z\phi+\frac{4}{3}ab'(4\partial_\sigma^4A_1-16\partial_\sigma^2A_1). \tag{42}$$

Then one gets

$$0=\frac{1}{\pi Gl}\left\{\sqrt{1+\frac{kl^2}{3R^2}+\frac{l^2c^2}{24R^8}}-1\right\}R^4+8b', \tag{43}$$

$$0=-\frac{c}{8\pi G}+32ab'. \tag{44}$$

Note that for minimal spinor coupling the second equation does not have a solution. Equation (44) can be solved with respect to c,

$$c=32\times 8\pi Gab', \tag{45}$$

but the boundary value ϕ_0 of ϕ becomes a free parameter.

We should also note that in the classical case that $b'=0$, there is no solution for Eqs. (43) and (44). From Eq. (44), we find $c=0$ if $b'=0$. Then, if we put $c=0$ and $b'=0$ in Eq. (43), there is no solution.

When the dilaton vanishes $(c=0)$ and the brane is the unit sphere $(k=3)$, Eq. (43) reproduces the result of Ref. [6] for $\mathcal{N}=4$ SU(N) super-Yang-Mills theory in the case of the large-N limit where b' is replaced by $-N^2/4(4\pi)^2$:

$$\frac{R^3}{l^3}\sqrt{1+\frac{R^2}{l^2}}=\frac{R^4}{l^4}+\frac{GN^2}{8\pi l^3}. \tag{46}$$

Let us define a function $F(R,c)$ as

$$F(R,c)\equiv \frac{1}{\pi G l}\left(\sqrt{1+\frac{kl^2}{3R^2}+\frac{l^2c^2}{24R^8}}-1\right)R^4, \tag{47}$$

which appears on the right-hand side (RHS) in (43).

For the $k>0$ case, $F(R,c)$ has a minimum at $R=R_0$, where R_0 is defined by

$$0=\frac{8kl^2}{3R_0^2}+\frac{k^2l^4}{R_0^4}-\frac{2l^2c^2}{3R_0^8}. \tag{48}$$

When $k>0$, there is only one solution for R_0. Therefore $F(R,c)$ in the case where $k>0$ (sphere case) is a monotonically increasing function of R when $R>R_0$ and a decreasing function when $R<R_0$. Since $F(R,c)$ is clearly a monotonically increasing function of c, we find for the $k>0$ and $b'<0$ case that R decreases when c increases if $R>R_0$; that is, the nontrivial dilaton makes the radius smaller. We can also find that there is no solution for R in Eq. (43) for very large $|c|$.

We can consider the $k<0$ case. When $c=0$, there is no solution for R in Eq. (43). We can find, however, that there is a solution if $|c|$ is large enough:

$$\frac{|c|}{\pi G\sqrt{24}}>-8b'. \tag{49}$$

Hence, for a constant bulk potential there is the possibility of the quantum creation of a 4D de Sitter or a 4D hyperbolic brane residing in 5D AdS bulk space. This occurs even for not exactly conformal invariant quantum brane matter. This finishes our consideration of quantum induced dilatonic brane worlds in the Einstein frame.

IV. QUANTUM INDUCED DILATONIC BRANE WORLDS IN THE STRING FRAME

We now transform the brane-world action in the Einstein frame [see Eq. (11)] into the Jordan frame. If we consider the scale transformation

$$g_{\mu\nu}\to e^{\rho}g_{\mu\nu}, \tag{50}$$

with the choice

$$e^{(D/2-1)\rho}=\alpha\phi \quad (\alpha \text{ is a constant}), \tag{51}$$

we find that the actions (12), (13), and (14) are transformed as

$$S_{\rm EH}=\frac{1}{16\pi G}\int d^5x\sqrt{g_{(5)}}\left[\alpha\phi R_{(5)}+\frac{4\alpha}{3\phi}\nabla_\mu\phi\nabla^\mu\phi -\frac{\alpha}{2}\phi\nabla_\mu\phi\nabla^\mu\phi+\left(\frac{12}{l^2}+\Phi(\phi)\right)(\alpha\phi)^{5/3}\right], \tag{52}$$

$$S_{\rm GH}=\frac{1}{8\pi G}\int d^4x\,\alpha\phi\sqrt{g_{(4)}}\nabla_\mu n^\mu, \tag{53}$$

$$S_1=-\frac{1}{16\pi G}\int d^4x(\alpha\phi)^{4/3}\sqrt{g_{(4)}}\left(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\right). \tag{54}$$

A. Bulk solution in the string frame

In the bulk, the variation over ϕ gives the following equation of motion:

$$0=\alpha R_{(5)}-\frac{4\alpha}{3\phi^2}\partial_\mu\phi\partial^\mu\phi-\frac{\alpha}{2}\partial_\mu\phi\,\partial^\mu\phi+\frac{5}{3}\left(\frac{12}{l^2}+\Phi(\phi)\right)\times\alpha^{5/3}\phi^{2/3}+\Phi'(\phi)(\alpha\phi)^{5/3}-\frac{8\alpha}{3}\nabla_\mu\left(\frac{1}{\phi}\partial^\mu\phi\right)+\alpha\nabla_\mu(\phi\,\partial^\mu\phi). \tag{55}$$

On the other hand, the variation over the metric $g^{\mu\nu}$ gives

$$0=-\frac{1}{2}\left[\alpha\phi R_{(5)}+\frac{4\alpha}{3\phi}\partial_\mu\phi\,\partial^\mu\phi-\frac{\alpha}{2}\phi\partial_\mu\phi\,\partial^\mu\phi+\left(\frac{12}{l^2}+\Phi(\phi)\right)(\alpha\phi)^{5/3}\right]g_{(5)\mu\nu}+\alpha\phi R_{(5)\mu\nu}-\alpha\nabla_\mu\partial_\nu\phi+\alpha g_{(5)\mu\nu}\Box\phi+\frac{4\alpha}{3\phi}\partial_\mu\phi\,\partial_\nu\phi-\frac{\alpha}{2}\partial_\mu\phi\,\partial_\nu\phi. \tag{56}$$

Thus, one gets the bulk equations of motion in the string frame. Using Eq. (56), we have

$$0=-\frac{3}{2}\left(\alpha\phi R_{(5)}+\frac{4\alpha}{3\phi}\partial_\mu\phi\,\partial^\mu\phi-\frac{\alpha}{2}\phi\partial_\mu\phi\,\partial^\mu\phi\right)-\frac{5}{2}\left(\frac{12}{l^2}+\Phi(\phi)\right)(\alpha\phi)^{5/3}+4\alpha\Box\phi. \tag{57}$$

Substituting Eq. (57) into Eqs. (55) and (56), one obtains

$$0=\alpha\nabla_\mu(\phi\partial^\mu\phi)+\Phi'(\phi)(\alpha\phi)^{5/3} \tag{58}$$

$$0=-\alpha\nabla_\mu\partial_\nu\phi-\frac{\alpha}{3}g_{(5)\mu\nu}\Box\phi+\alpha\phi R_{(5)\mu\nu}+\frac{1}{3}\left(\frac{12}{l^2}+\Phi(\phi)\right)\times(\alpha\phi)^{5/3}g_{(5)\mu\nu}+\frac{4\alpha}{3\phi}\partial_\mu\phi\,\partial_\nu\phi-\frac{\alpha}{2}\phi\partial_\mu\phi\,\partial_\nu\phi. \tag{59}$$

First, let us consider the $\Phi(\phi)=0$ case. In the Einstein frame, the solution is given by Eq. (23). The metric $g^{\rm J}_{(5)\mu\nu}$ in the Jordan frame is obtained with the help of Eqs. (50) and (51) or, more explicitly,

$$g^{\rm J}_{(5)\mu\nu}dx^\mu dx^\nu=(\alpha\phi)^{-2/3}\left(f(y)dy^2+y\sum_{i,j=0}^{3}\hat g_{ij}(x^k)dx^i dx^j\right),$$

$$f=\frac{l^2}{4y^2\left(1+\frac{c^2l^2}{24y^4}+\frac{kl^2}{3y}\right)}, \tag{60}$$

$$\phi = c\int dy \frac{l}{2\sqrt{y^6\left(1+\frac{c^2l^2}{24y^4}+\frac{kl^2}{3y}\right)}}.$$

One can check directly that the metric (60) satisfies Eqs. (58) and (59). Although the classical bulk solution in the Einstein frame is equivalent to the one in the Jordan frame, the physical interpretation of the spacetime is changed due to the factor of $(\alpha\phi)^{-2/3}$. Since the transformation is conformal, the causal structure of the spacetime is not changed, especially the situation that there is a curvature singularity at $y=0$ that is not changed. When $y\rightarrow\infty$, however, the spacetime is not asymptotically AdS but the metric behaves as

$$g^{\mathrm{J}}_{(5)\mu\nu}dx^\mu dx^\nu \sim \left(-\frac{\alpha c l}{4}\right)^{-2/3}\left(\frac{l^2}{4y^{2/3}}dy^2+y^{7/3}\sum_{i,j=0}^{3}\hat{g}_{ij}(x^k)dx^i dx^j\right). \tag{61}$$

If one defines a coordinate z by

$$z \equiv \left(-\frac{\alpha c l}{4}\right)^{-1/3}\frac{3l}{4}y^{2/3}, \tag{62}$$

the metric in Eq. (61) is rewritten by

$$g^{\mathrm{J}}_{(5)\mu\nu}dx^\mu dx^\nu \sim dz^2+\left(-\frac{\alpha c l}{4}\right)^{1/2}(4z/3l)^{7/2}\times\sum_{i,j=0}^{3}\hat{g}_{ij}(x^k)dx^i dx^j. \tag{63}$$

Then the warp factor behaves as a power of z, instead of the exponential function in the Einstein frame.

One can also consider the case that the dilaton potential $12/l^2+\Phi(\phi)$ is given by Eq. (27). Using the relations (50) and (51) between the Einstein frame and the Jordan frame, from Eqs. (26) and (28), we find the following solution:

$$g^{\mathrm{J}}_{(5)\mu\nu}dx^\mu dx^\nu = [\pm\alpha\sqrt{6}\ln(m^2y)]^{-2/3}\times\left(\frac{1}{-\frac{2ky}{9}+\frac{f_0}{y^2}}dy^2+y\sum_{i,j=0}^{3}\hat{g}_{ij}(x^k)dx^i dx^j\right)$$

$$\phi = \pm\sqrt{6}\ln(m^2y). \tag{64}$$

One can again check that the above solution satisfies Eqs. (58) and (59). Then the above result is equivalent to that in the Einstein frame. Comparing the obtained metric with that in the Einstein frame in Eqs. (26) and (28), there appears the factor of the logarithmic function of y, coming from the conformal transformation. In other words, the interpretation of the lengths in both frames is different while the solutions are equivalent.

B. Brane solutions in the string frame

Having proof of the explicit equivalency of bulk solutions, one can analyze the brane. From the actions in Eqs. (52), (53), and (54), the variation over ϕ gives the following equation on the boundary:

$$0=\frac{l^4 e^{4A}}{8\pi G}\left\{\left(\frac{8\alpha}{3\phi_0}-\alpha\phi_0\right)\partial_z\phi+8\alpha\partial_z A-\frac{4\alpha}{3}\left(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\right)\times(\alpha\phi_0)^{1/3}-\frac{l}{4}\Phi'(\phi)(\alpha\phi)^{4/3}\right\}. \tag{65}$$

Here we choose the metric as in Eq. (19) and ϕ_0 is the value of ϕ on the boundary. The variation over A gives the following equation:

$$0=\frac{48l^4}{16\pi G}e^{4A}\left[\alpha\phi_0\partial_z A+\frac{\alpha}{3}\partial_z\phi-\frac{1}{6}\left(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\right)(\alpha\phi_0)^{4/3}\right]. \tag{66}$$

The coordinate z and A in the warp factor are related to those in the Einstein frame, z_E and A_E, by

$$dz_E=(\alpha\phi)^{1/3}dz,\quad A_E=A+\frac{1}{3}\ln(\alpha\phi). \tag{67}$$

Then Eqs. (65) and (66) are rewritten as

$$0=\frac{l^4 e^{4A_E}}{8\pi G}\left\{-\partial_{z_E}\phi+\alpha(\alpha\phi_0)\times\left[8\partial_{z_E}A_E-\frac{4\alpha}{3}\left(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\right)-\frac{l}{4}\Phi'(\phi)\right]\right\}, \tag{68}$$

$$0=\frac{48l^4}{16\pi G}e^{4A_E}\left\{\partial_{z_E}A_E-\frac{1}{6}\left(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\right)\right\}. \tag{69}$$

Combining Eqs. (68) and (69), we obtain

$$0=\frac{l^4 e^{4A_E}}{8\pi G}\left\{-\partial_{z_E}\phi-\frac{l}{4}\Phi'(\phi)\right\}. \tag{70}$$

The obtained equations (69) and (70) are identical to the corresponding equations (31) and (32) without the quantum correction, respectively.

Choosing the metric of five-dimensional space-time as in Eq. (19),

$$ds^2=dz^2+e^{2A(z,\sigma)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu,$$

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu \equiv l^2(d\sigma^2+d\Omega_3^2), \tag{71}$$

where $d\Omega_3^2$ corresponds to the metric of three-dimensional unit sphere, we now include the quantum correction as in Eq. (36):

$$W=b\int d^4x\sqrt{\tilde{g}}\tilde{F}A+b'\int d^4x\Big\{A\Big[2\tilde{\Box}^2+\tilde{R}_{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu-\frac{4}{3}\tilde{R}\tilde{\Box}^2+\frac{2}{3}(\tilde{\nabla}^\mu\tilde{R})\tilde{\nabla}_\mu\Big]A+\Big(\tilde{G}-\frac{2}{3}\tilde{\Box}\tilde{R}\Big)A\Big\}-\frac{1}{12}\Big\{b''+\frac{2}{3}(b+b')\Big\}\times\int d^4x[\tilde{R}-6\tilde{\Box}A-6(\tilde{\nabla}_\mu A)(\tilde{\nabla}^\mu A)]^2. \tag{72}$$

Note that as typically in the Jordan frame there is no nonminimal dilaton coupling with matter we took minimal spinors, i.e., $a=0$. Then one obtains the following brane equations [instead of Eqs. (65) and (66)]:

$$0=\frac{l^4 e^{4A}}{8\pi G}\Big\{\Big(\frac{8\alpha}{3\phi_0}-\alpha\phi_0\Big)\partial_z\phi+8\alpha\partial_z A-\frac{4\alpha}{3}\Big(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\Big)\times(\alpha\phi_0)^{1/3}-\frac{l}{4}\Phi'(\phi)(\alpha\phi)^{4/3}\Big\}+b'(4\partial_\sigma^4 A-16\partial_\sigma^2 A)-4(b+b')[\partial_\sigma^4 A+2\partial_\sigma^2 A-6(\partial_\sigma A)^2\partial_\sigma^2 A], \tag{73}$$

$$0=\frac{48l^4}{16\pi G}e^{4A}\Big[\alpha\phi_0\partial_z A+\frac{\alpha}{3}\partial_z\phi-\frac{1}{6}\Big(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\Big)(\alpha\phi_0)^{4/3}\Big]+\frac{4}{3}ab'(4\partial_\sigma^4 A-16\partial_\sigma^2 A). \tag{74}$$

For $\Phi(\phi)=0$ case, substituting the solution in Eq. (60), one finds

$$0=\frac{1}{\pi Gl}\left\{\sqrt{1+\frac{kl^2}{3(\alpha\phi_0)^{2/3}R^2}+\frac{l^2c^2}{24(\alpha\phi_0)^{8/3}R^8}}-1\right\}\times(\alpha\phi_0)^{4/3}R^4+8b', \tag{75}$$

$$0=-\frac{c}{8\pi G}+\frac{1}{\pi Gl\phi_0}\times\left\{\sqrt{1+\frac{kl^2}{3(\alpha\phi_0)^{2/3}R^2}+\frac{l^2c^2}{24(\alpha\phi_0)^{8/3}R^8}}-1\right\}\times(\alpha\phi_0)^{4/3}R^4. \tag{76}$$

Combining Eqs. (75) and (76), one gets

$$0=-\frac{c}{8\pi G}-\frac{8b'}{\phi_0}. \tag{77}$$

Equation (77) has a nontrivial solution and can be solved with respect to ϕ_0:

$$\phi_0=-\frac{64\pi Gb'}{c}. \tag{78}$$

In the classical case that $b'=0$, there is no solution for Eq. (75). Let us define a function $F(R,c)$ as

$$F(R,c)\equiv\frac{1}{\pi Gl}\left\{\sqrt{1+\frac{kl^2}{3(\alpha\phi_0)^{2/3}R^2}+\frac{l^2c^2}{24(\alpha\phi_0)^{8/3}R^8}}-1\right\}\times(\alpha\phi_0)^{4/3}R^4. \tag{79}$$

It appears on the RHS side in Eq. (75).

For the $k>0$ case, $F(R,c)$ has a minimum at $R=R_0$, where R_0 is defined by

$$0=\frac{8kl^2}{3(\alpha\phi_0)^{2/3}R_0^2}+\frac{k^2l^4}{(\alpha\phi_0)^{4/3}R_0^4}-\frac{2l^2c^2}{3(\alpha\phi_0)^{8/3}R_0^8}. \tag{80}$$

When $k>0$, there is only one solution for R_0. Therefore $F(R,c)$ in the case of $k>0$ (sphere case) is a monotonically increasing function of R when $R>R_0$ and a decreasing function when $R<R_0$. Since $F(R,c)$ is clearly a monotonically increasing function of c, we find for the $k>0$ and $b'<0$ cases that R decreases when c increases if $R>R_0$; that is, the nontrivial dilaton makes the radius smaller.

Since one finds

$$F(R_0,c)=\frac{kl(\alpha\phi_0)^{2/3}R_0^2}{4\pi G}, \tag{81}$$

using Eqs. (79) and (80), Eq. (75) has a solution if

$$\frac{kl(\alpha\phi_0)^{2/3}R_0^2}{4\pi G}\leqslant-8b'. \tag{82}$$

That puts again some bounds on the dilaton value. When $|c|$ is small, using Eq. (80), one obtains

$$R_0^4\sim\frac{2c^2(\alpha\phi_0)^{-4/3}}{3k^2l^2},\quad F(R_0,c)\sim\frac{1}{4\pi G}\frac{|c|}{\sqrt{3}}. \tag{83}$$

Therefore Eq. (82) is satisfied for small $|c|$. On the other hand, when c is large, we get

$$R_0^6\sim\frac{c^2(\alpha\phi_0)^{-6/3}}{4k},\quad F(R_0,c)\sim\frac{(k|c|)^{2/3}}{4^{4/3}\pi G}. \tag{84}$$

Therefore Eq. (82) is not always satisfied and we have no solution for R in Eq. (43) for very large $|c|$.

We now consider the $k<0$ case. When $c=0$, there is no solution for R in Eq. (75). Let us define another function $G(R,c)$ as follows:

$$G(R,c)\equiv1+\frac{l^2c^2}{24(\alpha\phi_0)^{8/3}R^8}+\frac{kl^2}{3(\alpha\phi_0)^{2/3}R^2}. \tag{85}$$

Since $G(R,c)$ appears in the root of $F(R,c)$ in Eq. (79), $G(R,c)$ must be positive. Then, since

$$\frac{\partial G(R,c)}{\partial R}=-\frac{l^2c^2}{3(\alpha\phi_0)^{8/3}R^9}-\frac{2kl^2}{3(\alpha\phi_0)^{2/3}R^3}, \tag{86}$$

$G(R,c)$ has a minimum

$$1+\frac{kl^2}{4}\left(-\frac{2k}{c^2}\right)^{1/3}, \tag{87}$$

when

$$R^6=-\frac{c^2(\alpha\phi_0)^{-6/3}}{2k}. \tag{88}$$

Therefore, if

$$c^2 \geqslant \frac{k^4 l^6}{32}, \tag{89}$$

$F(R,c)$ is real for any positive value of R. Since

$$F(0,c)=\frac{|c|}{\pi G\sqrt{24}}, \tag{90}$$

and when $R\to\infty$,

$$F(R,c)\to\frac{kl(\alpha\phi_0)^{2/3}R^2}{6\pi G}<0, \tag{91}$$

there is a solution R in Eq. (75) if

$$\frac{|c|}{\pi G\sqrt{24}}>-8b'. \tag{92}$$

This is the same bound as in the Einstein frame (previous section).

Thus we demonstrated the complete equivalency of quantum induced inflationary (hyperbolic) dilatonic brane worlds in the Einstein and string (Jordan) frames.

Note that Eq. (75) is identical to the corresponding equation (43) in the Einstein frame if we regard $(\alpha\phi_0)^{1/3}R$ as the radius R_E in the Einstein frame:

$$R=(\alpha\phi_0)^{-1/3}R_E. \tag{93}$$

Then the solution has properties similar to those in the Einstein frame. Since b' is an order N quantity from Eq. (38), Equations (78) and (93) might tell us that the radius R in the Jordan frame is much smaller than the radius R_E in the Einstein frame if N is large. In the case that the brane is a sphere, the brane becomes de Sitter space. Since the rate of the expansion is given by $1/R$ in de Sitter space, the rate might become much larger if compared with that in the Einstein frame when N is large. Thus, even having a formal equivalency, the physical interpretation of the results obtained in the Jordan and Einstein frames may be different.

V. BRANE-WORLD BLACK HOLES IN THE STRING AND EINSTEIN FRAMES

In analogy with the Randall-Sundrum model [1], we now consider the following classical action of gravity coupled with dilaton ϕ in the Einstein frame with Lorentzian signature:

$$S=\frac{1}{16\pi G}\left[\int d^5x\sqrt{-g_{(5)}}\left(R_{(5)}-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-V(\phi)\right)-\sum_{i=\mathrm{hid,vis}}\int_{B_i}d^4x\sqrt{-g_{(4)}}U_i(\phi)\right]. \tag{94}$$

Here B_{hid} and B_{vis} are branes corresponding to hidden and visible sectors, respectively, and $U_i(\phi)$ corresponds to the vacuum energies on the branes in [1]. One assumes that $U(\phi)$ is dilaton dependent and its form is explicitly given later on from consistency of the equations of motion. The dilaton potential $V(\phi)$ is often given in terms of the superpotential $W(\phi)$:

$$V=\left(\frac{\partial W}{\partial\phi}\right)^2-\frac{4}{6}W^2. \tag{95}$$

We assume again that ϕ only depends on z and the metric has the following form:

$$ds^2=dz^2+e^{2A(z)}\tilde{g}_{ij}\,dx^i\,dx^j. \tag{96}$$

Here $\tilde{g}_{ij}$ is the metric of the Einstein manifold. We also suppose that the hidden and visible branes sit on $z=z_{\mathrm{hid}}$ and $z=z_{\mathrm{vis}}$, respectively. Then the equations of motion are given by

$$\phi''+4A'\phi'=\frac{\partial V}{\partial\phi}+\sum_{i=\mathrm{hid,vis}}\frac{\partial U_i(\phi)}{\partial\phi}\delta(z-z_i), \tag{97}$$

$$4A''+4(A')^2+\frac{1}{2}(\phi')^2=-\frac{1}{3}V(\phi)-\frac{2}{3}\sum_{i=\mathrm{hid,vis}}U_i(\phi)\delta(z-z_i), \tag{98}$$

$$A''+4(A')^2=ke^{-2A}-\frac{1}{3}V(\phi)-\frac{1}{6}\sum_{i=\mathrm{hid,vis}}U_i(\phi)\delta(z-z_i). \tag{99}$$

Here $'\equiv d/dz$. Especially, when $k=0$, Eqs. (97)–(99) have the following first integrals in the bulk:

$$\phi'=\sqrt{2}\,\frac{\partial W}{\partial\phi},\quad A'=-\frac{1}{3\sqrt{2}}W. \tag{100}$$

Near the branes, Eqs. (97)–(99) have the following form:

$$\phi''\sim\frac{\partial U_i(\phi)}{\partial\phi}\delta(z-z_i),\quad A''\sim-\frac{U_i(\phi)}{6}\delta(z-z_i) \tag{101}$$

or

$$2\phi'\sim\frac{\partial U_i(\phi)}{\partial\phi},\quad 2A'\sim-\frac{U_i(\phi)}{6}, \tag{102}$$

at $z=z_i$. Comparing Eqs. (102) with Eqs. (100), we find

$$U_{\rm hid}(\phi)=2\sqrt{2}W(\phi), \quad U_{\rm vis}(\phi)=-2\sqrt{2}W(\phi). \tag{103}$$

We should note that $k=0$ does not always mean that the brane is flat. As is well known, the Einstein equations are given by

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{1}{2}\Lambda g_{\mu\nu}=T^{\rm matter}_{\mu\nu}. \tag{104}$$

Here $T^{\rm matter}_{\mu\nu}$ is the energy-momentum tensor of the matter fields. If we consider the vacuum solution where $T^{\rm matter}_{\mu\nu}=0$, Eq. (104) can be rewritten as

$$R_{\mu\nu}=\frac{\Lambda}{2}g_{\mu\nu}. \tag{105}$$

If we put $\Lambda=2k$, Eq. (105) is nothing but an equation for the Einstein manifold. The Einstein manifolds are not always homogeneous manifolds like flat Minkowski, (anti–)de Sitter space

$$ds_4^2=-V(r)dt^2+V^{-1}(r)dr^2+r^2\,d\Omega^2, \quad V(r)=1-\frac{\Lambda}{6}r^2, \tag{106}$$

or Nariai space

$$ds_4^2=\frac{1}{\Lambda}(\sin^2\chi\,d\psi^2-d\chi^2-d\Omega^2), \tag{107}$$

but they can be some black hole solutions like Schwarzschild–(anti–)de Sitter black holes:

$$ds_4^2=-V(r)dt^2+V^{-1}(r)dr^2+r^2d\Omega^2,$$
$$V(r)=1-\frac{\tilde{G}_4M}{r}-\frac{\Lambda}{6}r^2. \tag{108}$$

As a special case, one can also consider a $k=0$ solution like Schwarzschild black holes:

$$ds_4^2\equiv\tilde{g}_{ij}\,dx^i\,dx^j$$
$$=-\left(1-\frac{\tilde{G}_4M}{r}\right)dt^2+\frac{dr^2}{\left(1-\frac{\tilde{G}_4M}{r}\right)}+r^2\,d\Omega^2. \tag{109}$$

In Eqs. (108) and (109), M is the mass of the black hole on the brane and the effective gravitational constant G_4 on the three-brane (here $d=4$) is given by

$$\frac{1}{G_4}=\frac{1}{G}\int_{z_{\rm hid}}^{z_{\rm vis}}dz\,e^{(d-2)A}. \tag{110}$$

In these solutions, the curvature singularity at $r=0$ has the form of a line penetrating the bulk 5D universe and the horizon makes a tube surrounding the singularity. The singularity and the horizon connect the hidden and visible branes. These black holes have been discussed in Ref. [13].

We now consider the Jordan frame in order to see if the singularity supports (or breaks) equivalency on the classical level. Using the scale transformation given by Eqs. (50) and (51) with $D=5$, the action (94) is rewritten as

$$S=\frac{1}{16\pi G}\int d^5x\sqrt{g_{(5)}}\left(\alpha\phi R_{(5)}+\frac{4\alpha}{3\phi}\nabla_\mu\phi\,\nabla^\mu\phi\right.$$
$$\left.-\frac{\alpha}{2}\phi\nabla_\mu\phi\,\nabla^\mu\phi-V(\phi)(\alpha\phi)^{5/3}\right)$$
$$-\sum_{i=\rm hid,vis}\int_{B_i}d^4x\sqrt{-g_{(4)}}(\alpha\phi)^{4/3}U_i(\phi). \tag{111}$$

Then, if we choose the metric as in Eq. (96) in the Jordan frame and ϕ only depends on z again, we obtain the following equations instead of Eqs. (97), (98), and (99):

$$\alpha[\phi\phi''+4A'\phi\phi'+(\phi')^2]$$
$$=\frac{\partial V}{\partial\phi}(\alpha\phi)^{5/3}+\sum_{i=\rm hid,vis}\frac{\partial U_i(\phi)}{\partial\phi}(\alpha\phi)^{4/3}\delta(z-z_i), \tag{112}$$

$$\alpha\phi[4A''+4(A')^2]+\frac{\alpha}{2}\phi(\phi')^2-\frac{4\alpha}{3\phi}(\phi')^2$$
$$+\frac{4\alpha}{3}(\phi''+A'\phi')$$
$$=-\frac{1}{3}V(\phi)(\alpha\phi)^{5/3}-\frac{2}{3}\sum_{i=\rm hid,vis}U_i(\phi)(\alpha\phi)^{4/3}\delta(z-z_i), \tag{113}$$

$$\alpha\phi[A''+4(A')^2]+\frac{\alpha}{3}(\phi''+7A'\phi')$$
$$=k\alpha\phi e^{-2A}-\frac{1}{3}V(\phi)(\alpha\phi)^{5/3}$$
$$-\frac{1}{6}\sum_{i=\rm hid,vis}U_i(\phi)(\alpha\phi)^{4/3}\delta(z-z_i). \tag{114}$$

If one transforms the above equations into those in the Einstein frame by changing

$$A\to A-\frac{1}{3}\ln(\alpha\phi), \tag{115}$$

$$dz\to(\alpha\phi)^{-1/3}dz$$
$$\left(\begin{array}{l} '\equiv\partial_z\to(\alpha\phi)^{1/3}\partial_z \\ ''=\partial_z^2\to(\alpha\phi)^{2/3}\left(\partial_z^2+\frac{\partial_z\phi}{3\phi}\partial_z\right)\end{array}\right),$$

then Eqs. (97), (98), and (99), which are the corresponding equations in the Einstein frame, are reproduced. Thus we can confirm the equivalence between the Jordan frame and the

Einstein frame description of dilatonic brane-world black holes on the classical level. Their physical interpretation may be again different.

VI. DISCUSSION

In summary, we discussed AdS/CFT induced quantum dilatonic brane worlds where branes may be a flat, de Sitter (inflationary) or anti–de Sitter universe. Actually, such objects appear in frames of the AdS/CFT correspondence [4] as warped compactification of relevant holographic renormalization group flow [5,6]. The role of the free parameter (brane tension) is played by the effective brane tension produced by the conformal anomaly of QFT sitting on the brane. Hence, only brane quantum effects are considered. We compared the construction of such quantum dilatonic brane worlds in two frames: the string and Einstein frames. A very nice feature of brane worlds is discovered: in all the examples under consideration the string and Einstein frames are equivalent. This holds true also for the number of classical dilatonic brane-world black holes. This is completely different from the case of quantum corrected 4D dilatonic gravity (Sec. II) where a de Sitter universe with a decaying dilaton exists in the Einstein frame but does not exist in the Jordan frame.

Quantum effects may be useful in other aspects of brane worlds. In particular, for flat branes bulk quantum effects (Casimir force) may be estimated [16–18] and used for radion stabilization. Unfortunately, in the usual Randall-Sundrum universe such quantum effects actually support the radion destabilization. Nevertheless, in the case of the thermal Randall-Sundrum scenario [19] such quantum effects may not only stabilize the radion but also may provide the necessary mass hierarchy [19] (at least, for some temperatures). It would be extremely interesting to estimate the bulk quantum effects for dilatonic backgrounds and to understand their role (as well as the frame dependence of such a Casimir effect) in the creation of dilatonic brane worlds.

Another interesting line of research is related to an account of quantum effects on graviton perturbations around the brane. As is demonstrated in a previous section, they may modify the massive graviton modes around the hyperbolic brane. Clearly, in other regimes for the quantum induced dilatonic (asymptotically) AdS brane more complicated dynamics may be expected.

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APPENDIX: REMARKS ON GRAVITATIONAL PERTURBATIONS AROUND THE HYPERBOLIC BRANE

In [14,15], AdS_4 branes in AdS_5 were discussed and the existence of a massive normalizable mode of the graviton was found. In these papers, the tensions of the branes are free parameters, but in the case treated in the present paper, the tension is dynamically determined.

Let us study the role of a dynamically generated tension in obtaining massive graviton modes. Moreover, we consider a dilatonic brane world. We now regard the brane as an object with a tension $U(\phi)$ and assume that the brane can be effectively described by the following action:

$$S_{\rm brane}=-\frac{1}{16\pi G}\int d^4x\sqrt{-g_{(4)}}U(\phi). \tag{A1}$$

If one assumes a metric in the form of Eq. (19), then using the Einstein equation, we find

$$\partial_z^2 A+4(\partial_z A)^2=k{\rm e}^{-2A}+\frac{4}{l^2}+\frac{\Phi(\phi)}{3}-\frac{U(\phi)}{6}\delta(z-z_0). \tag{A2}$$

Then, at $z=z_0$,

$$\partial_z A\big|_{z=z_0}=-\frac{U(\phi)}{12}. \tag{A3}$$

For simplicity, we consider the case of the constant dilaton potential $\Phi(\phi)=0$. Comparing Eq. (A3) with Eqs. (41) and (43), one gets

$$U(\phi)=-\frac{12}{l}+\frac{96\pi G b'}{R^4}. \tag{A4}$$

We should note that the tension becomes R dependent due to the quantum correction. In the case of the AdS brane $k<0$, if no dilaton is included, the boundary equation (43) does not have any solution for R. When there is a nontrivial dilaton and the parameter c is large enough, Eq. (43) has a solution. If c is very large,

$$R^4\sim\frac{c}{\pi G}+8b'. \tag{A5}$$

We now consider the perturbation by assuming the metric in the following form:

$$ds^2={\rm e}^{2\hat{A}(\zeta)}[d\zeta^2+(\hat{g}_{\mu\nu}+{\rm e}^{-3\hat{A}(\zeta)/2}h_{\mu\nu})dx^\mu\, dx^\nu]. \tag{A6}$$

By choosing the gauge conditions $h_\mu{}^\mu=0$ and $\nabla^\mu h_{\mu\nu}=0$, one obtains the following equation:

$$\left(-\partial_\zeta^2+\frac{9}{4}(\partial_\zeta\hat{A})^2+\frac{3}{2}\partial_\zeta^2\hat{A}\right)h_{\mu\nu}=m^2h_{\mu\nu}. \tag{A7}$$

Here m^2 corresponds to the mass of the graviton on the brane:

$$\left(\hat{\Box}\pm\frac{1}{R^2}\right)h_{\mu\nu}=m^2h_{\mu\nu}. \tag{A8}$$

Here $\hat{\Box}$ is a four-dimensional d'Alembertian constructed on $\hat{g}_{\mu\nu}$ and the $+$ $(-)$ sign corresponds to the (anti–)de Sitter brane. Since $-e^{A}\,d\zeta = dz = \sqrt{f}dy$ and $e^{A} = \sqrt{y}/l$, we find, especially for the case of a constant dilaton potential,

$$\zeta = -\int dy\sqrt{\frac{f(y)}{y}} = -\frac{l^2}{2}\int \frac{dy}{\sqrt{y^3\left(1+\frac{c^2l^2}{24y^4}+\frac{kl^2}{3y}\right)}}. \tag{A9}$$

We now consider the case where c is very large; then,

$$f(y)\sim\frac{6y^2}{c^2}. \tag{A10}$$

Since $y_0 = R^2$, if there is a brane at $y = y_0$, Eq. (A5) can be rewritten as

$$y_0^2 \sim \frac{c}{\pi G} + 8b'. \tag{A11}$$

If we choose $\zeta = 0$ when $y = y_0$, Eqs. (A9) and (A10) give

$$|\zeta| = -\frac{1}{|c|}\sqrt{\frac{8}{3}}y^{3/2} + \zeta_0, \quad \zeta_0 \equiv \frac{1}{|c|}\sqrt{\frac{8}{3}}y_0^{3/2} > 0. \tag{A12}$$

Note that the brane separates two bulk regions corresponding to $\zeta < 0$ and $\zeta > 0$, respectively. Since y takes the value in $[0, y_0]$, ζ takes the value in $[-\zeta_0, \zeta_0]$. Since $A = \frac{1}{2}\ln y$, from Eq. (A7), one gets

$$\left(-\partial_\zeta^2 - \frac{1}{4(|\zeta|-\zeta_0)^2} - \frac{1}{\zeta_0}\delta(\zeta)\right)h_{\mu\nu} = m^2 h_{\mu\nu}. \tag{A13}$$

The zero-mode solution with m^2 of Eq. (A13) is given by

$$h_{\mu\nu} = \sqrt{\zeta_0 - |\zeta|}. \tag{A14}$$

The general solution of Eq. (A13) with $m^2 \neq 0$ is given by the Bessel functions

$$h_{\mu\nu} = aJ_0[m(\zeta_0 - |\zeta|)] + bN_0[m(\zeta_0 - |\zeta|)]. \tag{A15}$$

The coefficients a and b are constants of integration, and they are determined to satisfy the boundary condition

$$\left.\frac{\partial_\zeta h_{\mu\nu}}{h_{\mu\nu}}\right|_{\zeta\to 0+} = -\frac{1}{2\zeta_0}. \tag{A16}$$

Note that the zero-mode solution (A14) satisfies this boundary condition (A16). If $b \neq 0$, the solution in Eq. (A15) diverges at $\zeta = \pm\zeta_0$ and would not be normalizable. If $b = 0$, the condition (A16) reduces to

$$J_1(m\zeta_0) = 0, \tag{A17}$$

that is,

$$m\zeta_0 = 0, 3.8317\ldots, 7.0155\ldots, \ldots. \tag{A18}$$

The nonvanishing solutions for m^2 give the mass of the massive graviton modes. Thus, these results indicate that 4D dilatonic gravity on the quantum induced hyperbolic brane may be trapped near the brane.

Since ζ_0 is given by y_0 in Eq. (A12) and y_0 is expressed by Eq. (A11), with the help of b', which comes from the quantum correction and is negative, the quantum correction makes ζ_0 smaller and increases the massive graviton mode mass m. It would be of interest to discuss graviton or dilaton perturbations around asymptotically hyperbolic branes in other regimes and to compare the corresponding predictions in different frames.

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